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Uniqueness in Restricted Range Approximation with Betweenness

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Let X be a compact Hausdorff space. Let $\mathscr{G}$ be a subset of $C(X)$ with the betweenness property [2]. Let u and v be continuous into the extended real line, $u < v$. We consider uniqueness of best Chebyshev approximations under the constraint on approximations G that

$$u \leqslant G \leqslant v.$$

This approximation problem was first considered in [3].

DEFINITION. $\mathscr{G}$ has *zero-sign compatibility* if, for any two distinct elements G, H, any closed set Z of zeros of $G - H$, and any continuous function s taking the values $+1$ or -1 on Z, there exists $F \in \mathscr{G}$ such that

$$\operatorname{sgn}(F(x) - G(x)) = s(x), \qquad x \in Z.$$

Zero-sign compatibility is necessary for uniqueness in ordinary best Chebyshev approximation and sufficient for uniqueness in ordinary best Chebyshev approximation if $\mathscr{G}$ has the betweenness property [2].

DEFINITION. A normal topological space in which each closed set is a countable intersection of open sets is called perfectly normal.

The perfectly normal spaces include all subsets of finite dimensional Euclidean space.

THEOREM. *Let X be perfectly normal. If $\mathscr{G}$ has zero-sign compatibility, best restricted range approximations are unique for f in $[u, v]$.*

Proof. Suppose H and I are distinct best approximations to f. By

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arguments of Lemmas 3, 5 of [2], by taking G in the λ-set of H, I for any λ in (0, 1), we get G also best and G, H agree on $\hat{M}(G)$, where

$$\hat{M}(G) = \{x: |f(x) - G(x)| = \|f - G\|\} \bigcup \{x: u(x) = G(x)\}$$

$$\bigcup \{x: v(x) = G(x)\}.$$

Define

$$\begin{aligned} s(x) &= \operatorname{sgn}(f(x) - G(x)) & \quad |f(x) - G(x)| &= \|f - G\| \\ &= +1 & G(x) &= u(x) \\ &= -1 & G(x) &= v(x). \end{aligned}$$

No inconsistency can arise in the above definition. Suppose, for example, we had $f(x) - G(x) = -\|f - G\| < 0$ and $G(x) = u(x)$, then $f(x) < u(x)$ and $f \notin [u, v]$.

Now x such that $s(x) = -1$ and x such that $s(x) = +1$ form disjoint closed sets. By a result of Dugundji [1, p. 148], there is a continuous extension of s to X such that $|s(x)| < 1$ for all other x. By the definition of zero-sign compatibility, there is $F \in \mathscr{G}$ with

$$\operatorname{sgn}(F(x) - G(x)) = s(x) \qquad x \in \hat{M}(G).$$

But this contradicts Theorem 2 of [3].

COROLLARY. *Let X be perfectly normal. If best approximations are unique in ordinary Chebyshev approximation, they are unique in the restricted range problem if $f \in [u, v]$.*

In the case of one-sided approximation from above, with $u = f$, $v = +\infty$, an explicit extension for s is available and we do not need to assume perfect normality. Choose

$$s(x) = 1 - 2\,|f(x) - G(x)|/\|f - G\|$$

then $-1 \leqslant s \leqslant 1$ with equality only if $G(x) - f(x) = \|f - G\|$ or $G(x) = f(x)$. One-sided approximation from below is handled similarly.

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